Blackwell-type Theorems for Weighted Renewal Functions[1]

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Abstract

For a numerical sequence $\{a_n\}$ satisfying broad assumptions on its "behaviour on average" and a random walk $S_n = \xi_1 + \cdots + \xi_n$ with i.i.d. jumps ξ_j with positive mean μ, we establish the asymptotic behaviour of the sums

$$\sum_{n\geqslant 1} a_n \mathbf{P}(S_n \in [x, x+\Delta)) \quad \text{as} \quad x \to \infty,$$

where $\Delta > 0$ is fixed. The novelty of our results is not only in much broader conditions on the weights $\{a_n\}$, but also in that neither the jumps ξ_j nor the weights a_j need to be positive. The key tools in the proofs are integro-local limit theorems and large deviation bounds. For the jump distribution F, we consider conditions of four types: (a) the second moment of ξ_j is finite, (b) F belongs to the domain of attraction of a stable law, (c) the tails of F belong to the class of the so-called locally regularly varying functions, (d) F satisfies the moment Cramér condition. Regarding the weights, in cases (a)–(c) we assume that $\{a_n\}$ is a so-called ψ-locally constant on average sequence, $\psi(n)$ being the scaling factor ensuring convergence of the distributions of $(S_n - \mu n)/\psi(n)$ to the respective stable law. In case (d) we consider sequences of weights of the form $a_n = b_n e^{qn}$, where $\{b_n\}$ has the properties assumed about the sequence $\{a_n\}$ in cases (a)–(c) for $\psi(n) = \sqrt{n}$.

Key words and phrases: weighted renewal function, Blackwell renewal theorem, Gnedenko–Stone–Shepp theorems, integro-local theorems, large deviation probabilities, locally constant functions, regular variation.

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1 Introduction

Consider the following motivating problem where we will meet the objects to be studied in the present paper. Let $\{\xi_j\}_{j\geqslant 1}$ and $\{\tau_j\}_{j\geqslant 0}$ be independent sequences of random variables (r.v.'s), $\xi_j \stackrel{d}{=} \xi$ being independent identically distributed with a common distribution function F and finite mean $\mu := \mathbf{E}\,\xi > 0$, $\tau_j > 0$ being arbitrary with $a_j := \mathbf{E}\,\tau_j < \infty$. Set

$$S_n := \sum_{j=1}^n \xi_j, \qquad T_n := \sum_{j=0}^n \tau_j, \qquad n = 0, 1, 2, \ldots,$$

and consider the generalised renewal process $S_{\nu(t)}$, where $\nu(t) := \inf\{n \geqslant 0 : T_n > t\}$, $t \geqslant 0$. Clearly, the process describes the movement of a particle that rests at zero during the time interval $[0, T_0)$, then changes its location at time T_0 by a jump of size ξ_1 and stays at the new location during the time interval $[T_0, T_1)$, then jumps by ξ_2, and so on. Processes of that kind are often encountered in applications, e.g. in queueing.

The time spent by the process $S_{\nu(t)}$ in a given interval

$$\Delta[x) := [x, x+\Delta), \qquad x \in \mathbb{R}, \quad \Delta > 0,$$

is equal to

$$\tau(\Delta[x)) := \sum_{n=0}^\infty \tau_n \mathbf{1}(S_n \in \Delta[x)),$$

where $\mathbf{1}(A)$ is the indicator of the event A. It is often important to know the behaviour of the expectation

$$h(x, \Delta) := \mathbf{E}\,\tau(\Delta[x)) = \sum_{n=0}^\infty a_n \mathbf{P}(S_n \in \Delta[x))$$

of that time and, in particular, its asymptotics as $x \to \infty$.

Observe also that, if we "swap time with space" and assume that $\xi \geqslant 0$, then $\tau(\Delta[x))$ will be the increments of the generalised renewal process $T_{\eta(x)}$ on the intervals $\Delta[x)$, $\eta(x)$ being the renewal process generated by the sequence $\{\xi_j\}$. In that case, the assumption $\tau_j > 0$ (so that $a_j > 0$) is not needed. In what follows, we will be allowing negative values for a_n.

When the series

$$H(x) := \sum_{n=0}^\infty a_n \mathbf{P}(S_n < x)$$

converges, its sum $H(x)$ is referred to as the weighted renewal function for the sequence $\{\xi_j\}$. In that case, $h(x, \Delta) = H(x+\Delta) - H(x)$ is just the increment of the

function H on the interval $\Delta[x)$. Note, however, that $h(x,\Delta)$ can exist even when $H(x)$ does not. Indeed, if the weights a_n are bounded then a sufficient condition for $H(x)$ to exist and be finite is that $\mathbf{E}\,(\xi^-)^2 < \infty$, where $v^- := \min\{v, 0\}$, while if $a_n \equiv 1$ and $\mathbf{E}\,(\xi^-)^2 = \infty$, then $H(x) = \infty$ (see e.g. § 10.1 in [4]). On the other hand, the quantity $h(x,\Delta)$ is always finite provided that $0 < \mu < \infty$, $|a_n| \leqslant C < \infty$ (this follows from the transience of the random walk $\{S_n\}$, see e.g. Section VI.10 in [10]).

There is substantial literature devoted to studying the asymptotics of $H(x)$ and $h(x,\Delta)$ as $x \to \infty$, recent surveys of the area and some further references being available in [20, 15]. In what follows, we will mostly be mentioning results referring to $h(x,\Delta)$.

In the special case $a_n = 1/n$, $H(x)$ is called the harmonic renewal function. It is closely related to the concepts of factorization identities, ladder epochs and heights etc. That case was dealt with in [13, 12, 1, 3].

In the case of regularly varying function (r.v.f.) $a_x = x^\gamma L(x)$ (here L is slowly varying), it was shown in [2] that, if $\gamma \geqslant 0$ and a_x is ultimately increasing, then the condition

$$\mathbf{E}\,(\xi^-)^2 a_{\xi^-} < \infty \tag{1.1}$$

is necessary and sufficient for having $H(x) < \infty$ for all x (thus extending the above-mentioned results on the finiteness of H) which, in its turn, is equivalent to the asymptotics

$$H(x) \sim \frac{x}{\mu(\gamma+1)}\, a_{x/\mu}, \qquad x \to \infty, \tag{1.2}$$

while if $\gamma \in (-1, 0]$ and a_x is ultimately decreasing, relation (1.1) still implies (1.2). Here and in what follows, $\sim$ denotes asymptotic equivalence: we write $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \to \infty$. We will also use the convention that $b_x := b_{\lfloor x \rfloor}$ for a sequence $\{b_n\}$, where $\lfloor x \rfloor$ is the integer part of x.

For the function $h(x,\Delta)$, it was shown in [2] that, in the non-lattice case, if a_x is an ultimately increasing r.v.f. of index $\gamma \geqslant 0$, then condition

$$\mathbf{E}\,\xi^- a_{\xi^-} < \infty \tag{1.3}$$

is equivalent to the asymptotics

$$h(x,\Delta) \sim \frac{\Delta}{\mu}\, a_{x/\mu}, \qquad x \to \infty \tag{1.4}$$

(with an analog of the relation holding true in the arithmetic case), and that (1.4) also holds when $\gamma \in (-1, 0]$ and a_x is ultimately decreasing (in that case (1.3) is always met in the finite mean case). Earlier research for that case was done in [14] (for $a_x = x^\gamma$)

and [16, 8].

Clearly, (1.4) and its analog in the arithmetic case are direct extensions of the celebrated Blackwell theorem in Renewal Theory that describes the asymptotics of $h(x,\Delta)$ as $x\to\infty$ in the case where $a_n\equiv 1$.

The special case $a_n=n!\binom{n+k-1}{k}$ was investigated in [21], where three-term approximations where obtained for $H(x)$ for nonlattice F with a finite second moment.

In [20] it was shown that, in the non-lattice case, if the sequence a_n is non-decreasing with $\lim_{n\to\infty} a_n=\infty$ and

$$\lim_{s\downarrow 0}\limsup_{x\to\infty}\frac{a_{x(1+s)}}{a_x}=1,$$

then (1.4) holds true. That paper also showed that (1.4) is true in the case where $a_n\downarrow 0$ as $n\to\infty$ (under some additional conditions).

When the sequence a_n is fast changing, the asymptotics of $h(x,\Delta)$ will be different from (1.4). The special case where $a_n=A^n$ for a fixed $A>0$, $\mathbf{P}(\xi\geqslant 0)=1$ and F has an absolutely continuous component or is arithmetic, was considered in [19]. It was shown there, in particular, that, provided that F has finite moment generating function and A belongs to a suitable range of values, the asymptotics of $h(x,\Delta)$ will also be of exponential nature (the precise form of the asymptotics is given in our Section 4 below, see relations (4.1), (4.2) with $b_n\equiv 1$).

Of the known general results, the following assertion stated as Theorem 6.1 in [15] is the closest one to the main topic of the present paper: *If the sequence $\{a_n\}$ is regularly oscillating in the sense that*

$$\lim_{x,y\to\infty,\ x/y\to 1}\frac{a_x}{a_y}=1 \tag{1.5}$$

and, as $x\to\infty$, one has

$$F_+(x):=\mathbf{P}(\xi\geqslant x)=o\left(\frac{a_x}{A_x}\right),\quad \textit{where}\quad A_n:=\sum_{j=0}^{n}a_j, \tag{1.6}$$

then, in the non-lattice case, for any fixed $\Delta>0$ one has (1.4).

A relation analogous to (1.4) holds under the same assumptions in the arithmetic case as well (Theorem 3.1 in [15]). In fact, it were these results that drew our attention to the problem on the asymptotic behaviour of $h(x,\Delta)$ as $x\to\infty$, as we realized that the conditions imposed in [15] on the weight sequence $\{a_n\}$ could be substantially relaxed provided that F belongs to the domain of attraction of a stable law.

In the present paper, we establish asymptotics of the form (1.4) under much more general conditions on $\{a_n\}$ than have previously been dealt with and allowing ξ to assume values of both signs. In Section 2 we consider two cases that can be treated using the same approach based on the integro-local Gnedenko–Stone–Shepp theorems

and some large deviation bounds from [5]: the case of finite variance, and the case where F belongs to the domain of attraction of a stable law with index $\alpha \in (1, 2)$. In these cases, using the fact that, owing to the integro-local theorems, the sequence of probability values $\mathbf{P}(S_n \in [x, x+\Delta))$ varies in a nice regular way, we demonstrate (1.4) under the assumption that the weight sequence is "ψ-locally constant on average" (see the definition of that property in Section 2) and is non-decreasing or non-increasing "on average". Note that, under the assumptions made with regard to $\{a_n\}$, the above conditions on the distribution of ξ are close to the ones necessary for (1.4). In Section 3 we obtain necessary and sufficient conditions for (1.4) to hold under the assumption that $\mathbf{E}\,\xi^2 < \infty$ and the tails of F are of locally regular variation (see the definition thereof in the beginning of Section 3). In Section 4 we find the asymptotics of $h(x, \Delta)$ under the assumptions that F satisfies the moment Cramér condtion and $a_n = b_n e^{qn}$, where $\{b_n\}$ is a ψ-locally constant sequence with $\psi(n) = \sqrt{n}$, $q = \text{const} \neq 0$.

2 The case where the second moment is finite or there is convergence to a non-normal stable law

To formulate appropriate conditions on the weight sequence, first recall the definition of an asymptotically ψ-locally constant (ψ-l.c.) function (see Definition 1.2.7 in [5]). Let $\psi(t) > 1$, $t > 0$, be a fixed non-decreasing function.

A function $g(x) > 0$ is said to be ψ-l.c. if, for any fixed $v \in \mathbb{R}$ such that $x + v\psi(x) \geqslant cx$ for some $c > 0$ and all large enough x, one has

$$\lim_{x\to\infty} \frac{g(x + v\psi(x))}{g(x)} = 1.$$

A sequence $\{u_n\}$ is called ψ-l.c. if $g(x) := u_x$ is a ψ-l.c. function. Everywhere in what follows ψ will be assumed to be an r.v.f.

Note that ψ-l.c. functions with $\psi \equiv 1$ are sometime referred to as "long-tailed". Furthermore, it follows from Theorem 1 in [6] that, under broad assumptions on ψ, a ψ-l.c. function will, in the terminology of [11], be "h-insensitive" (or "h-flat") with $h \equiv \psi$ (see Definition 2.18 in [11]).

Introduce the following conditions on the weight sequence.

Condition $[\boldsymbol{\psi}]$ is satisfied for sequence $\{a_n\}$ if *there exists an r.v.f. $d(t)$ such that $d(t) = o(\psi(t))$ as $t \to \infty$ and the "averaged sequence"*

$$\tilde{a}_n := \frac{1}{d(n)} \sum_{n \leqslant k < n+d(n)} a_k > 0 \tag{2.1}$$

is ψ-l.c. One can always assume that the averaging interval length $d(n)$ is integer-valued.

Sequences satisfying this condition we will call "ψ-locally constant on average". In the present section, the function ψ will be chosen to be the scaling from the respective limit theorem for the sequence of partial sums S_n.

Any ψ-l.c. sequence clearly is a ψ-l.c. on average sequence: it satisfies condition $[\boldsymbol{\psi}]$ with $d(n) \equiv 1$. A simple example of a ψ-l.c. on average sequence that is not ψ-l.c. is provided by a periodic sequence $a_n = a_{n-\lfloor n/d \rfloor d}$ with a period $d \geqslant 2$, such that there are distinct values among $a_0, a_1, \dots, a_{d-1}$. In that case, $\tilde{a}_n = \text{const}$, and $d(n)$ can be chosen to be a (constant) multiple of d. Note that condition $[\boldsymbol{\psi}]$ does not exclude the case where some of the weights a_n can be negative.

Sequences satisfying one of the following two conditions could be called "monotone on average".

Condition $[\boldsymbol{\psi}, \boldsymbol{\downarrow}]$ is met for $\{a_n\}$ if that sequence *satisfies* $[\boldsymbol{\psi}]$ *and, for some* $r > 1$ *and* $c < \infty$, *one has* $|a_k| \leqslant c\tilde{a}_n$ *for all* $k > n/r$.

Condition $[\boldsymbol{\psi}, \boldsymbol{\uparrow}]$ is met for $\{a_n\}$ if that sequence *satisfies* $[\boldsymbol{\psi}]$ *and, for some* $r > 1$ *and* $c < \infty$, *one has* $|a_k| \leqslant c\tilde{a}_n$ *for all* $k < nr$.

By $[\boldsymbol{\psi}, \uparrow \cup \downarrow]$ we will denote the condition that at least one of conditions $[\boldsymbol{\psi}, \boldsymbol{\downarrow}]$ and $[\boldsymbol{\psi}, \boldsymbol{\uparrow}]$ is satisfied for the sequence in question.

We will also need the following conditions on the tails

$$F_+(t) = \mathbf{P}(\xi \geqslant t) \quad \text{and} \quad F_-(t) := \mathbf{P}(\xi < -t)$$

of the jump distribution. We will assume that either

$$\sigma^2 := \mathbf{E}\,(\xi - \mu)^2 < \infty \tag{2.2}$$

or the next condition is met:

Condition $[\mathbf{R}_{\alpha,\rho}]$. *The two-sided tail*

$$F_*(t) := F_-(t) + F_+(t)$$

is an r.v.f. with index $-\alpha$, $\alpha \in (0, 2)$, and there exists the limit

$$\lim_{t \to \infty} \frac{F_+(t)}{F_*(t)} =: \frac{1}{2}(\rho + 1) \in [0, 1].$$

Given that condition $[\mathbf{R}_{\alpha,\rho}]$ is satisfied, we put

$$b(t) := \inf\{x > 0 : F^*(x) < 1/t\}, \quad t > 1 \tag{2.3}$$

(recall that $b(t) = t^{1/\alpha} l(t)$, where $l(t)$ is slowly varying as $t \to \infty$, see e.g. Theorem 1.1.4(v) in [5]).

Now let

$$\psi(t) := \begin{cases} \sigma\sqrt{t} & \text{if (2.2) is met,} \\ b(t) & \text{if } [\mathbf{R}_{\alpha,\rho}] \text{ is met.} \end{cases} \tag{2.4}$$

It is well-known that, given that either (2.2) or $[\mathbf{R}_{\alpha,\rho}]$ is satisfied, the sequence of the distributions of the scaled partial sums $(S_n - \mu n)/\psi(n)$ converges weakly as $n \to \infty$ to the respective stable law that we will denote by Φ (see e.g. Theorem 1.5.1 in [5]).

Moreover, the integro-local Stone–Shepp theorem holds true as well: *if F is non-lattice then, for the approximation error $\epsilon_n(x, \Delta)$ in the representation*

$$\mathbf{P}(S_n \in \Delta[x)) = \frac{\Delta}{\psi(n)}\,\phi\left(\frac{x - \mu n}{\psi(n)}\right) + \frac{\epsilon_n(x, \Delta)}{\psi(n)}, \tag{2.5}$$

ϕ being the density of Φ, one has

$$\lim_{n\to\infty} \sup_{\Delta\in[\Delta_1,\Delta_2]} \sup_x \left|\epsilon_n(x, \Delta)\right| = 0 \tag{2.6}$$

for any fixed $0 < \Delta_1 < \Delta_2 < \infty$ (see e.g. Theorems 8.7.1 and 8.8.2 in [4], or Theorem 6.1.2 in [5]).

In the arithmetic maximum span 1 case (where $\mathbf{P}(\xi \in \mathbb{Z}) = 1$ and g.c.d.$\{k_1 - k_2 : \mathbf{P}(\xi = k_1)\mathbf{P}(\xi = k_2) > 0\} = 1$) an analog of the above theorem holds for probabilities $\mathbf{P}(S_n = k)$, $k \in \mathbb{Z}$ (Gnedenko's theorem, see e.g. Theorems 8.7.3 and 8.8.4 in [4], or Theorem 6.1.1 in [5]).

Now we will state the main results of this section.

Theorem 2.1 *Let the jump distribution F be non-lattice and condition $[\boldsymbol{\psi}, \downarrow]$ be satisfied for $\{a_n\}$. Furthermore, assume that one of the following two conditions is met: either*

(i) $\mathbf{E}\,\xi^2 < \infty$ *and the right tail F_+ of F admits a regularly varying majorant:*

$$F_+(t) \leqslant V(t) := t^{-\alpha} L_V(t), \qquad t > 0, \tag{2.7}$$

where $\alpha > 2$ and $L_V(t)$ is slowly varying as $t \to \infty$, such that

$$V(x) = o\left(\frac{\tilde{a}_x}{\widetilde{A}_x}\right) \quad as \quad x \to \infty, \qquad \widetilde{A}_n := \sum_{k \leqslant n} \tilde{a}_k, \tag{2.8}$$

or

(ii) *one has*

$$[\mathbf{R}_{\alpha,\rho}] \textit{ holds for some } \alpha \in (1,2), \qquad F_+(x) = o\left(\frac{\tilde{a}_x}{\widetilde{A}_x}\right) \ \ as \ \ x \to \infty. \tag{2.9}$$

Then, for any fixed $0 < \Delta_1 < \Delta_2 < \infty$, relation

$$h(x, \Delta) \sim \frac{\Delta}{\mu}\, \tilde{a}_{x/\mu}, \qquad x \to \infty, \tag{2.10}$$

holds uniformly in $\Delta \in [\Delta_1, \Delta_2]$.

Remark 2.1 We will write $f(x) \preccurlyeq g(x)$ as $x \to \infty$ if $f(x) \leqslant cg(x)$ for some $c > 0$ and all large enough x. If one has

$$A_x \succcurlyeq \overline{A}_x := \sum_{k \leqslant x} |a_k| \quad \text{as} \quad x \to \infty, \tag{2.11}$$

(this condition is close to $[\boldsymbol{\psi}, \downarrow]$ and is always met when $a_k \geqslant 0$), then one can replace (2.8) with

$$V(x) = o\left(\frac{\tilde{a}_x}{A_x}\right) \quad \text{as} \quad x \to \infty. \tag{2.12}$$

Indeed, if (2.11) holds then, for $y = x + d(x)$, we have

$$A_y \succcurlyeq \overline{A}_y \geqslant \widetilde{\overline{A}}_x \geqslant \widetilde{A}_x, \quad \text{where} \quad \widetilde{\overline{A}}_x := \sum_{n \leqslant x} \widetilde{\overline{a}}_n, \quad \widetilde{\overline{a}}_n := \frac{1}{d(n)} \sum_{n \leqslant k < n + d(n)} |a_k|.$$

Therefore, assuming $[\boldsymbol{\psi}]$ and (2.12) satisfied, one has

$$V(x) \preccurlyeq V(y) = o\left(\frac{\tilde{a}_y}{A_y}\right) = o\left(\frac{\tilde{a}_x}{A_y}\right) = o\left(\frac{\tilde{a}_x}{\widetilde{A}_x}\right),$$

where the first relation holds since V is an r.v.f. and $x \sim y$, and the third one follows from the fact that $\{\tilde{a}_x\}$ is a ψ-l.c. Thus (2.8) is established. It is not hard to see that condition $V(x) = o\big(\tilde{a}_x / \overline{A}_x\big)$ is also sufficient for (2.8).

Theorem 2.2 *Let the jump distribution F be non-lattice and condition $[\boldsymbol{\psi}, \uparrow]$ be satisfied for $\{a_n\}$. Furthermore, assume that one of the following two conditions is met: either*

(i) $\mathbf{E}\,\xi^2 < \infty$ *and the left tail* F_- *of* F *admits a regularly varying majorant:*

$$F_-(t) \leqslant W(t) := t^{-\beta} L_W(t), \qquad t > 0, \tag{2.13}$$

where $\beta > 2$ *and* $L_W(t)$ *is slowly varying as* $t \to \infty$*, such that*

$$\sum_{n \geqslant 0} \tilde{a}_n W(n) < \infty, \tag{2.14}$$

or

(ii) *one has*

$$[\mathbf{R}_{\alpha,\rho}] \text{ holds for some } \alpha \in (1,2), \qquad \sum_{n \geqslant 0} \tilde{a}_n F_-(n) < \infty. \tag{2.15}$$

Then, for any fixed $0 < \Delta_1 < \Delta_2 < \infty$*, relation* (2.10) *holds uniformly in* $\Delta \in [\Delta_1, \Delta_2]$.

Remark 2.2 If $a_n \geqslant 0$ then, in conditions (2.14) and (2.15), one can replace $\tilde{a}_n$ with a_n. Indeed, since $d(t)$ and $W(t)$ are r.v.f.'s, adding up the coefficients of a_n in relation (2.14) we obtain

$$\sum_{n \geqslant 0} \tilde{a}_n W(n) = \sum_{n \geqslant 0} a_n e_n W(n),$$

where $e_n \to 1$ as $n \to \infty$. It follows from here that the series $\sum \tilde{a}_n W(n)$ converges iff $\sum a_n e_n W(n)$ does. Also, it is not hard to see that convergence of $\sum |a_n| W(n)$ is sufficient for (2.14). Similar remarks apply to condition (2.15) as well.

Remark 2.3 If, instead of assuming that F is non-lattice, one assumes that F is arithmetic, then the assertions of Theorems 2.1 and 2.2 remain true provided that both x and Δ are integer. In other words, in the arithmetic case, for integer-valued $x \to \infty$ one will have

$$\sum_{n=0}^{\infty} a_n \mathbf{P}(S_n = x) \sim \frac{1}{\mu}\, \tilde{a}_{x/\mu}.$$

Proof of Theorems 2.1 and 2.2. Put

$$n_\pm := \frac{x}{\mu} \pm N\psi(x), \qquad m_\pm := \frac{x}{\mu}\, r^{\pm 1}, \tag{2.16}$$

where $N = N(x) \to \infty$ slowly enough as $x \to \infty$ (the choice of N will be discussed below), $r > 1$ is the quantity from conditions $[\boldsymbol{\psi}, \downarrow]$, $[\boldsymbol{\psi}, \uparrow]$.

Represent $h(x, \Delta)$ as

$$h(x, \Delta) = \Sigma_{2-} + \Sigma_{1-} + \Sigma_0 + \Sigma_{1+} + \Sigma_{2+}, \tag{2.17}$$

where the terms on the right-hand side of (2.17) are the sub-sums of the products $a_n\mathbf{P}(S_n \in \Delta[x))$ over the following ranges of n values:

$$\Sigma_{2-} := \sum_{n<m_-}, \ \Sigma_{1-} := \sum_{m_-\leqslant n<n_-}, \ \Sigma_0 := \sum_{n_-\leqslant n<n_+}, \ \Sigma_{1+} := \sum_{n_+\leqslant n<m_+}, \ \Sigma_{2+} := \sum_{n\geqslant m_+}.$$

Lemmata 2.1–2.5 are devoted to evaluating these subsums. The assertions of Theorems 2.1 and 2.2 will follow from these results.

Lemma 2.1 *If condition* $[\boldsymbol{\psi}]$ *is satisfied for the sequence* $\{a_n\}$ *and either condition* (2.2) *or condition* $[\mathbf{R}_{\alpha,\rho}]$ *with* $\alpha \in (1,2)$ *is met then*

$$\Sigma_0 \sim \frac{\Delta}{\mu}\,\tilde{a}_{x/\mu},$$

provided that the value $N = N(x)$ *in* (2.16) *tends to infinity slowly enough as* $x \to \infty$.

Proof. Assume for simplicity that n_- is integer (as we will see from what follows, changing the values of $n_\pm$ and $m_\pm$ by amounts of the order $o(\psi(x))$ does not change anything in the proof).

Construct a sequence n_k, $0 \leqslant k \leqslant K+1$, by setting $n_0 := n_-$,

$$n_{k+1} := n_k + d(n_k), \quad k = 0, 1, \ldots, K := \min\{k \geqslant 0 : \, n_{k+1} \geqslant n_+\}$$

(so that $K \sim 2N\psi(x)/d(x/\mu)$), and amend somewhat n_+ by putting, in agreement with the above remark, its value equal to $n_+ := n_{K+1}$. Partition the set $[n_-, n_+)$ into semi-intervals $[n_k, n_{k+1})$, $k = 0, 1, \ldots, K$. On each of these intervals, the probabilities

$$p_n := \mathbf{P}(S_n \in \Delta[x))$$

remain "almost constant" (in the ratio sense). More precisely, putting

$$\pi(n) := \phi\big((x-\mu n)/\psi(n)\big)/\psi(n),$$

we obtain by virtue of (2.5), (2.6) and the relation $d(n_k) = o(\psi(x/\mu))$ that, for $n \in [n_k, n_{k+1})$, one has

$$p_n = (1+o(1))p_{n_k} = (1+o(1))\Delta\pi(n_k)$$

uniformly in $k \in [0, K]$ provided that $N \to \infty$ slowly enough as $x \to \infty$. Hence the sub-sums $\sum_{n\in[n_k,n_{k+1})} a_n p_n$, $k = 0, 1, \ldots, K$, that comprise Σ_0 are of the form

$$(1+o(1))(n_{k+1}-n_k)\tilde{a}_{n_k}\Delta\pi(n_k).$$

But it follows from condition $[\boldsymbol{\psi}]$ that $\tilde a_{n_k} = (1+o(1))\tilde a_{x/\mu}$ uniformly in $k \leqslant K$ provided that $N \to \infty$ slowly enough as $x \to \infty$ (see Theorem 1 in [6]), so that

$$\sum_{n\in[n_k,n_{k+1})} a_n p_n = (1+o(1))\Delta \tilde a_{x/\mu}(n_{k+1}-n_k)\pi(n_k),$$

and therefore

$$\begin{aligned}\Sigma_0 &= (1+o(1))\Delta \tilde a_{x/\mu}\sum_{k=0}^{K}(n_{k+1}-n_k)\pi(n_k)\\ &= (1+o(1))\Delta \tilde a_{x/\mu}\sum_{k=0}^{K}\frac{n_{k+1}-n_k}{\psi(n_k)}\,\phi\biggl(\frac{x-\mu n_k}{\psi(n_k)}\biggr).\end{aligned} \tag{2.18}$$

As

$$\frac{n_{k+1}-n_k}{\psi(n_k)} \sim \frac{n_{k+1}-n_k}{\psi(x/\mu)} \to 0, \qquad \phi\biggl(\frac{x-\mu n_k}{\psi(n_k)}\biggr) \sim \phi\biggl(\frac{x-\mu n_k}{\psi(x/\mu)}\biggr)$$

uniformly in $k \leqslant K$ (if $N \to \infty$ slowly enough as $x \to \infty$), the last sum in (2.18) is (up to the factor $(1+o(1))$) a Riemann sum for the integral

$$\int_{x/\mu-N\psi(x)}^{x/\mu+N\psi(x)} \phi\biggl(\frac{x-\mu t}{\psi(x/\mu)}\biggr)\frac{dt}{\psi(x/\mu)} = \frac{1+o(1)}{\mu}\int_{-Nc}^{Nc}\phi(u)\,du \to \frac{1}{\mu},$$

where $c = \lim_{x\to\infty}\psi(x)/\psi(x/\mu) = \mu^{1/\alpha}$ ($\alpha = 2$ in the case where $\mathbf{E}\,\xi^2 < \infty$). The lemma is proved. $\square$

Lemma 2.2 *Under the conditions of Lemma 2.1, if* $[\boldsymbol{\psi},\uparrow\cup\downarrow]$ *is met then*

$$\Sigma_{1\pm} = o(\tilde a_{x/\mu}), \qquad x \to \infty.$$

Proof. By virtue of $[\boldsymbol{\psi},\uparrow\cup\downarrow]$ one has

$$|\Sigma_{1\pm}| \preccurlyeq \tilde a_{x/\mu} h_\pm(x), \quad \text{where} \quad h_-(x) := \sum_{m_-\leqslant n<n_-} p_n, \quad h_+(x) := \sum_{n_+\leqslant n<m_+} p_n.$$

Clearly, $h_\pm(x) \leqslant h(x) - h_0(x)$, where, as $x \to \infty$,

$$h(x) := \sum_{n\geqslant 0} p_n \to \frac{\Delta}{\mu} \tag{2.19}$$

by Blackwell's theorem, and

$$h_0(x) := \sum_{n_-\leqslant n<n_+} p_n \to \frac{\Delta}{\mu} \tag{2.20}$$

by Lemma 2.1 for the sequence $a_n \equiv 1$. It follows from here that $h_\pm(x) \to 0$ as $x \to \infty$. The lemma is proved. $\square$

To bound $\Sigma_{2\pm}$ we will need the following extension of Lemma 6.1 from [15] to the case where $\mathbf{P}(\xi < 0) > 0$. Set

$$\underline{S}^{(n)} := \inf_{k \geqslant n} (S_k - S_n) \stackrel{d}{=} \underline{S}^{(0)}, \qquad \overline{S}_n := \max_{k \leqslant n} S_k.$$

Since $\mu > 0$, the r.v. $\underline{S}^{(0)}$ is proper and

$$\gamma := \mathbf{P}(\underline{S}^{(0)} = 0) > 0 \tag{2.21}$$

(see e.g. § 12.2 in [4]).

Lemma 2.3 *If $\Delta > 0$ is such that $F_+(\Delta) > 0$ then, for any $n \geqslant 1$, one has*

$$\sum_{k \leqslant n} \mathbf{P}(S_k \in \Delta[x)) \leqslant \frac{\mathbf{P}(\overline{S}_n \geqslant x)}{\gamma F_+(\Delta)}, \tag{2.22}$$

$$\sum_{k \geqslant n} \mathbf{P}(S_k \in \Delta[x)) \leqslant \frac{\mathbf{P}(S_n + \underline{S}^{(n)} < x + \Delta)}{\gamma F_+(\Delta)}. \tag{2.23}$$

Proof. For any $I \subset \mathbb{R}_+$ holds

$$\sum_{k \in I} \mathbf{P}(S_k \in \Delta[x)) = \Sigma' + \Sigma'', \tag{2.24}$$

where

$$\Sigma' := \sum_{k \in I} \mathbf{P}\big(S_k \in \Delta[x),\ S_{k+j} \notin \Delta[x) \ \textit{for all}\ j \geqslant 1\big),$$
$$\Sigma'' := \sum_{k \in I} \mathbf{P}\big(S_k \in \Delta[x),\ S_{k+j} \in \Delta[x) \ \textit{for some}\ j \geqslant 1\big).$$

Since the events in the probabilities in the sum Σ' are mutually exclusive, one has

$$\Sigma' \leqslant \mathbf{P}\Big(\bigcup_{k \in I} \{S_k \in \Delta[x)\}\Big) \leqslant \begin{cases} \mathbf{P}(\overline{S}_n \geqslant x) & \text{if } I = (0, n], \\ \mathbf{P}(S_n + \underline{S}^{(n)} < x + \Delta) & \text{if } I = [n, \infty). \end{cases}$$

Further, as S_k and $\xi_{k+1}+\underline{S}^{(k+1)} \stackrel{d}{=} \xi_1+\underline{S}^{(1)}$ are independent of each other, we obtain

$$\begin{aligned}\Sigma'' &\leqslant \sum_{k\in I}\mathbf{P}\big(S_k\in\Delta[x),\, \inf_{j>0} S_{k+j}<x+\Delta\big)\\ &\leqslant \sum_{k\in I}\mathbf{P}(S_k\in\Delta[x),\, \xi_{k+1}+\underline{S}^{(k+1)}<\Delta)\\ &= \mathbf{P}(\xi_1+\underline{S}^{(1)}<\Delta)\sum_{k\in I}\mathbf{P}(S_k\in\Delta[x)),\end{aligned}$$

where, using definition (2.21) and the independence of ξ_1 and $\underline{S}^{(1)}$, we have

$$\begin{aligned}\mathbf{P}(\xi_1+\underline{S}^{(1)}<\Delta) &= \mathbf{P}(\xi_1+\underline{S}^{(1)}<\Delta,\, \underline{S}^{(1)}=0)+\mathbf{P}(\xi_1+\underline{S}^{(1)}<\Delta,\, \underline{S}^{(1)}<0)\\ &\leqslant \mathbf{P}(\xi_1<\Delta,\, \underline{S}^{(1)}=0)+\mathbf{P}(\underline{S}^{(1)}<0)\\ &= \mathbf{P}(\xi_1<\Delta)\gamma+1-\gamma = 1-\gamma F_+(\Delta).\end{aligned}$$

Substituting the bounds we established for Σ' and Σ'' into (2.24) first in the case where $I=(0,n]$ and then where $I=[n,\infty)$, we obtain the assertion of the lemma. □

Lemma 2.4 *For the relation*

$$\Sigma_{2-} = o(\tilde a_{x/\mu}), \qquad x\to\infty, \tag{2.25}$$

to hold it suffices that one of the following conditions is met:

(i) $\{a_n\}$ *satisfies* $[\boldsymbol{\psi},\downarrow]$ *and one of conditions* (i), (ii) *of Theorem* 2.1 *is met;*

(ii) $\{a_n\}$ *satisfies* $[\boldsymbol{\psi},\uparrow]$.

Proof. (i) Assume for simplicity (and without losing generality) that $x/\mu \equiv m_- r = r^{M+1}$ (see (2.16)) for some integer $M\geqslant 1$, where $r>1$ is the quantity from condition $[\boldsymbol{\psi},\downarrow]$, and consider the semi-intervals

$$I_j := [r^{j-1}, r^j), \qquad j=1,2,\dots \tag{2.26}$$

First let $\Delta>0$ be such that $F_+(\Delta)>0$. Then, using condition $[\boldsymbol{\psi},\downarrow]$, inequality (2.22) and notation $\chi_j := \mathbf{1}(I_j\cap\mathbb{N}\neq\varnothing)$, we obtain that

$$\begin{aligned}|\Sigma_{2-}| &\leqslant \sum_{j=1}^M\sum_{n\in I_j}|a_n|p_n \preccurlyeq \sum_{j=1}^M \tilde a_{r^j}\sum_{n\in I_j}p_n\\ &\leqslant \sum_{j=1}^M \tilde a_{r^j}\mathbf{P}(\overline{S}_{r^j}\geqslant x)\chi_j.\end{aligned} \tag{2.27}$$

Recall that, in the case of finite variance, if condition (2.7) is satisfied and $n \preccurlyeq y \to \infty$, then

$$\mathbf{P}\Big(\max_{k\leqslant n}(S_k - \mu k) \geqslant y\Big) \preccurlyeq nV(y) \tag{2.28}$$

(see Corollary 4.1.4(i) in [5]). Observing that

$$\{\overline{S}_{r^j} \geqslant x\} = \{\overline{S}_{r^j} - \mu r^j \geqslant x - \mu r^j\} \subset \Big\{\max_{k\leqslant r^j}(S_k - \mu k) \geqslant x - \mu r^j\Big\}$$

and $x - \mu r^j \geqslant x - \mu r^M = (1 - r^{-1})x$ for $j \leqslant M$, we obtain from (2.28) the bound

$$\mathbf{P}(\overline{S}_{r^j} \geqslant x) \leqslant \mathbf{P}\Big(\max_{n\leqslant r^j}(S_n - \mu n) \geqslant (1 - r^{-1})x\Big) \preccurlyeq r^j V(x), \quad j \leqslant M.$$

Substituting this bound into (2.27) and using the inequality $\tilde a_{r^j} \leqslant c\tilde a_n$, $n \in I_j$ (it holds by virtue of condition $[\boldsymbol{\psi}, \downarrow]$), we find that

$$|\Sigma_{2-}| \preccurlyeq V(x)\sum_{j=1}^{M} r^j \tilde a_{r^j}\chi_j \preccurlyeq V(x/\mu)\sum_{j=1}^{M}\sum_{n\in I_j}\tilde a_n \leqslant V(x/\mu)\tilde A_{x/\mu}.$$

Now relation (2.25) follows immediately from condition (2.8).

In the case of convergence to a non-normal stable law, the above argument remains valid provided that, when justifying inequality (2.28), we replace the reference to Corollary 4.1.4 in [5] with that to Corollary 3.1.2 from the same monograph.

If $F_+(\Delta) = 0$ (i.e. $\mathbf{P}(\xi < \Delta) = 1$) then one can always find a $k \in \mathbb{N}$ such that for $\Delta_k := \Delta/k$ one has $F_+(\Delta_k) > 0$. After that, it remains to apply the bound we have just derived to each of the terms on the right-hand side of the representation

$$h(x, \Delta) = h(x, k\Delta_k) = \sum_{j=0}^{k-1} h(x + j\Delta_k, \Delta_k).$$

(ii) In this case,

$$|\Sigma_{2-}| \leqslant \sum_{n<m_-} |a_n| p_n \preccurlyeq \tilde a_{x/\mu}\sum_{n<m_-} p_n \leqslant \tilde a_{x/\mu}(h(x) - h_0(x)) = o(\tilde a_{x/\mu})$$

by virtue of (2.19) and (2.20). The lemma is proved. □

Lemma 2.5 *For the relation*

$$\Sigma_{2+} = o(\tilde a_{x/\mu}), \qquad x \to \infty,$$

to hold it suffices that one of the following conditions is met:

(i) $\{a_n\}$ *satisfies* $[\boldsymbol{\psi},\downarrow]$*;*

(ii) $\{a_n\}$ *satisfies* $[\boldsymbol{\psi},\uparrow]$ *and one of conditions* (i), (ii) *of Theorem* 2.2 *is met.*

Proof. (i) In this case,

$$|\Sigma_{2+}| \preccurlyeq \tilde a_{x/\mu} \sum_{n>m_+} p_n \leqslant \tilde a_{x/\mu}(h(x)-h_0(x)) = o(\tilde a_{x/\mu})$$

by virtue of (2.19), (2.20).

(ii) Using notation (2.26) and assuming for simplicity that now $x/\mu \equiv m_+/r = r^{M-2}$ for some integer $M>1$, where $r>1$ is the quantity from condition $[\boldsymbol{\psi},\uparrow]$, we obtain from condition $[\boldsymbol{\psi},\uparrow]$ and inequality (2.23) that

$$\begin{aligned} |\Sigma_{2+}| &\leqslant \sum_{j\geqslant M}\sum_{n\in I_j} |a_n| p_n \preccurlyeq \sum_{j\geqslant M} \tilde a_{r^{j-1}} \sum_{n\in I_j} p_n \\ &\preccurlyeq \sum_{j\geqslant M} \tilde a_{r^{j-1}} \mathbf{P}\big(S_{r^{j-1}} + \underline{S}^{(r^{j-1})} < x+\Delta\big) \end{aligned} \tag{2.29}$$

(in contrast to the proof of Lemma 2.4, we do not need to use the indicators χ_j here since M is large and therefore $r^j - r^{j-1} > 1$ for $j \geqslant M$).

Note that, for $j\geqslant M$ and large enough M, the mean value $\mathbf{E}\, S_{r^{j-1}} = \mu r^{j-1} \equiv xr$ of the first r.v. in the probabilities on the right-hand side of (2.29) exceeds the value $x+\Delta$ by

$$\mu r^{j-1} - (x+\Delta) = \mu r^{j-1} - \mu r^{M-2} - \Delta \geqslant \mu r^{j-2}(r-1) - \Delta \geqslant 2cr^j$$

for some $c>0$, whereas $\underline{S}^{(r^{j-1})}$ is stochastically minorated by the global minimum $\underline{S}^0$. Therefore, to bound the probabilities of the right-hand side of (2.29), we will need to use large deviation results for the left tails of the distributions of the r.v.'s S_n and $\underline{S}^0$. In the case of finite variance, the required bound for the former r.v. is contained in the assertion of Corollary 4.1.4(i) in [5] (applied to the random walk with jumps $-\xi_j$). To bound the left tail of $\underline{S}^0$, it suffices to use condition (2.13) to construct a random walk with i.i.d. jumps that have a positive mean and distribution with a regularly varying left tail, such that it stochastically minorates $\{S_n\}$, and then make use of Theorem 7.5.1 in [5].

Following this approach, we obtain the bound

$$\begin{aligned} \mathbf{P}\big(S_{r^{j-1}} + \underline{S}^{(r^{j-1})} < x+\Delta\big) &\leqslant \mathbf{P}\big(S_{r^{j-1}} - \mu r^{j-1} < -cr^j\big) + \mathbf{P}\big(\underline{S}^{(0)} < -cr^j\big) \\ &\preccurlyeq r^{j-1} W(cr^{j-1}) + cr^j W(cr^j) \preccurlyeq r^j W(r^j). \end{aligned}$$

Now returning to (2.29), we obtain

$$|\Sigma_{2+}| \preccurlyeq \sum_{j\geqslant M} \tilde{a}_{r^{j-1}} r^j W(r^j) \preccurlyeq \sum_{j\geqslant M} W(r^j) \sum_{n\in I_j} \tilde{a}_n \preccurlyeq \sum_{n\geqslant m_+} \tilde{a}_n W(n),$$

where we again used condition $[\boldsymbol{\psi}, \uparrow]$ and the fact that W is an r.v.f. Therefore, under condition (2.14), one has

$$\Sigma_{2+} = o(1) = o(\tilde{a}_{x/\mu}),$$

where the last relation follows from $[\boldsymbol{\psi}, \uparrow]$.

In the case of convergence to a non-normal stable law, instead of Corollary 4.1.4 one should make use of Corollary 3.1.2 in [5]. The lemma is proved. □

The assertions of Theorems 2.1 2.2 follow from Lemmata 2.1, 2.2, 2.4 2.5. The required uniformity in Δ in the bounds established in Lemmata 2.2–2.5 is obvious.

3 The case where the tails of the jump distribution are of local regular variation

We will say that $V(x)$ is a *locally regularly varying function* (l.r.v.f.) if it is an r.v.f. and, moreover, for any fixed $\Delta > 0$ holds

$$V(x) - V(x+\Delta) = \Delta v(x)(1+o(1)), \qquad x \to \infty, \tag{3.1}$$

where $v(x) = \alpha V(x)/x$, $-\alpha$ is the index of the r.v.f. $V(x)$ (cf. (2.7)). It will be assumed throughout this section that $\alpha > 2$. Property (3.1) could be called the “differentiability of V at infinity”.

It is clear that (3.1) will hold if the slowly varying function L_V in the representation on the right-hand side of (2.7) is differentiable and $L'_V(x) = o(L_V(x)/x)$ as $x \to \infty$.

If the tail F_+ (or F_-) of the distribution F is an l.r.v.f., then the derivation of the asymptotics of $h(x,\Delta)$ as $x \to \infty$ substantially simplifies due to the fact that integro-local theorems valid on the whole half-line are available for such distributions. For instance, if F_+ is an l.r.v.f. then, as it was established in [17], in the case where $\mathbf{E}\,\xi = 0$ and $\sigma^2 = \mathbf{E}\,\xi^2 < \infty$, one has the following representation valid for $x > c\sqrt{n}$, $c = \mathrm{const} > 0$:

$$\mathbf{P}(S_n \in \Delta[x)) = \left[\frac{\Delta}{\sigma\sqrt{n}}\,\phi\left(\frac{x}{\sigma\sqrt{n}}\right) + n\Delta v(x)\right](1+o(1)), \qquad n \to \infty.$$

where ϕ is the standard normal density and

$$v(x) = \frac{\alpha F_+(x)}{x}$$

is the function from representation (3.1) for $V(x) = F_+(x)$. A similar assertion holds for $\mathbf{P}(S_n \in \Delta[x))$ in the range $x \leqslant -c\sqrt{n}$, provided that F_- is an l.r.v.f.; in that case, we will use notation

$$w(x) := \frac{\beta F_-(x)}{x},$$

where $-\beta$ is the index of the r.v.f. F_-.

From here and the Stone–Shepp theorem we immediately obtain the following assertions that reduces the problem of computing the asymptotics of $h(x, \Delta)$ to a purely analytic exercise.

Theorem 3.1 *Let $\mathbf{E}\,\xi = \mu > 0$, $\sigma^2 < \infty$, and the tails $F_+(x) = x^{-\alpha}L_V(x)$, $F_-(x) = x^{-\beta}L_W(x)$ be l.r.v.f.'s, $\min\{\alpha, \beta\} > 2$, $b(n) := \sigma\sqrt{n}$. Then, for any fixed $\Delta > 0$, $c_- \in (0, 1/\mu)$ and $c_+ \in (1/\mu, \infty)$, one has*

$$h(x, \Delta) = \Delta\biggl[\frac{1}{\sigma\sqrt{2\pi n}} \sum_{c_- x \leqslant n \leqslant c_+ x} a_n e^{-(x-\mu n)^2/(2\sigma^2 n)} + \sum_{n > x/\mu + b(x)} a_n n w(\mu n - x) + \sum_{n < x/\mu - b(x)} a_n n v(x - \mu n)\biggr](1 + o(1)) \qquad (3.2)$$

as $x \to \infty$.

The next assertion shows that, in Theorem 2.2(ii), condition (2.15) on the relationship between the tails F_- and weights a_n cannot be weakened, and that condition (2.9) on the relationship between a_n and F_+ can be extended to the minimal one under the assumptions of the present section. Since we assume that F has a finite second moment, we automatically put $\psi(t) := \sigma\sqrt{t}$ (cf. (2.4)).

Theorem 3.2 *Let $\mathbf{E}\,\xi = \mu > 0$, $\sigma^2 < \infty$, and $a_n \geqslant 0$.*

(i) *If F_- is an l.r.v.f. then the divergence of the series $\sum_n a_n F_-(n) = \infty$ implies that $h(x, \Delta) = \infty$ for any $\Delta > 0$.*

(ii) *If condition $[\boldsymbol{\psi}, \uparrow]$ is satisfied for $\{a_n\}$ and F_- is an l.r.v.f. then (2.10) holds iff the series $\sum_n a_n F_-(n)$ converges.*

(iii) *Let condition $[\boldsymbol{\psi}, \downarrow]$ be satisfied for $\{a_n\}$ and F_+ be an l.r.v.f. Then, as $x \to \infty$,*

$$h(x, \Delta) = \frac{\Delta}{\mu}\tilde{a}_{x/\mu}(1 + o(1)) + \Delta \sum_{n < x/(r\mu)} a_n n v(x - \mu n)(1 + o(1)). \qquad (3.3)$$

In this case, relation (2.10) *holds iff*

$$F_+(x) = o\left(\frac{x\tilde{a}_{x/\mu}}{B_{x/(r\mu)}}\right), \qquad x \to \infty, \tag{3.4}$$

where $B_x := \sum_{k \leqslant x} k a_k$.

Remark 3.1 Note that the principal part of the second term on the right-hand side of (3.3) is always between the values

$$\Delta B_{x/(r\mu)} v(x) \quad \text{and} \quad \Delta B_{x/(r\mu)} v((r-1)x).$$

Remark 3.2 If the series A_∞ diverges and a_x is an r.v.f., then conditions (3.4) and (2.8) are equivalent. However, if $A_\infty < \infty$ then condition (3.4) is broader than (1.6), (2.8). For example, if $B_\infty < \infty$ then $A_\infty < \infty$, condition (3.4) takes the form $F_+(x) = o(x\tilde{a}_{x/\mu})$, whereas condition (2.8) will mean that $F_+(x) = o(\tilde{a}_{x/\mu})$.

Proof of Theorem 3.2. (i) This assertion follows from the inequality

$$h(x, \Delta) \geqslant \Delta \sum_{n \geqslant x/(r\mu)} a_n n w(\mu n - x) \succcurlyeq \sum_{n \geqslant x/(r\mu)} a_n F_-(n).$$

(ii) We will make use of the representation

$$h(x, \Delta) = \Sigma_{1-} + \Sigma_0 + \Sigma_{1+} + \Sigma_{2+},$$

where

$$\Sigma_{1-} := \sum_{n < x/\mu - Nb(x)} a_n p_n, \qquad \Sigma_0 := \sum_{|n - x/\mu| \leqslant Nb(x)} a_n p_n,$$

$$\Sigma_{1+} := \sum_{n \in (x/\mu + Nb(x),\, x/\mu + c_1\sqrt{x \ln x}]} a_n p_n, \qquad \Sigma_{2+} := \sum_{n > x/\mu + c_1\sqrt{x \ln x}} a_n p_n.$$

By Lemma 2.1, provided that $N = N(x) \to \infty$ slowly enough as $x \to \infty$, one has

$$\Sigma_0 \sim \frac{\Delta}{\mu}\, \tilde{a}_{x/\mu},$$

whereas by Lemma 2.2

$$\Sigma_{1-} = o(\tilde{a}_{x/\mu}).$$

Further, for $c_1 = \text{const} > 0$ one has

$$\Sigma_{1+} = \frac{\Delta(1+o(1))}{\sigma\sqrt{2\pi n}} \sum_{n\in(x/\mu+Nb(x),\, x/\mu+c_1\sqrt{x\ln x}]} a_n e^{-(x-\mu n)^2/(2\sigma^2 n)}$$
$$\preccurlyeq \tilde a_{x/\mu} e^{-cN^2} = o(\tilde a_{x/\mu})$$

and

$$\Sigma_{2+} = (1+o(1))\Delta \sum_{n>x/\mu+c_1\sqrt{x\ln x}} a_n n w(\mu n - x). \tag{3.5}$$

If the series $\sum_n a_n F_-(n)$ diverges then $\Sigma_{2+} = \infty$ (see the proof of part (i) above). Otherwise make use of the representation

$$\Sigma_{2+} = \Sigma_{2,1+} + \Sigma_{2,2+}, \qquad \Sigma_{2,1+} := \sum_{n\in(x/\mu+c_1\sqrt{x\ln x},\, xr/\mu]}, \qquad \Sigma_{2,2+} := \sum_{n>xr/\mu}.$$

It is not hard to see that

$$\Sigma_{2,1+} \preccurlyeq \tilde a_{x/\mu} \sum_{n\in(x/\mu+c_1\sqrt{x\ln x},\, xr/\mu]} n w(\mu n - x) \preccurlyeq \tilde a_{x/\mu} x F_-(Nb(x)) = o(\tilde a_{x/\mu})$$

and, by virtue of convergence $\sum_n a_n F_-(n) < \infty$ and condition $[\boldsymbol{\psi}, \uparrow]$, we obtain

$$\Sigma_{2,2+} \preccurlyeq \sum_{n>xr/\mu} a_n F_-(n) = o(1) = o(\tilde a_{x/\mu}).$$

This proves assertion (ii).

(iii) Representation (3.3) is established using the same argument as the one employed to prove part (ii). That condition (3.4) is necessary and sufficient for (2.10) follows from Remark 3.1. $\square$

Remark 3.3 The case where the tails $F_\pm$ decay as semi-exponential functions can be dealt with in a similar fashion, as integro-local theorems for $\mathbf{P}(S_n \in \Delta[x))$ that are valid on the whole half-line are available in that case as well (see [7], [18]). The semi-exponential case is "intermediate" between the case where the tails are l.r.v.f.'s and the case of exponentially fast decaying tails that we will consider in the next section.

4 The case where the moment Cramér condition is met

Denote by $\varphi(\lambda) := \mathbf{E}\, e^{\lambda\xi}$ the moment generating function of ξ and set

$$\lambda_- := \inf\{\lambda : \varphi(\lambda) < \infty\} \leqslant 0, \qquad \lambda_+ := \sup\{\lambda : \varphi(\lambda) < \infty\} \geqslant 0.$$

We will assume in this section that the moment Cramér condition is satisfied:

$$\lambda_+ - \lambda_- > 0.$$

Denote by $\lambda_{\min}$ the minimum point of $\varphi(\lambda)$. Clearly, it will also be the minimum point of the (convex) function $L(\lambda) := \ln \varphi(\lambda)$ and, since $\mu > 0$, one always has $\lambda_{\min} < \lambda_+$. Indeed, if $\lambda_- < 0$ then $\lambda_{\min} < 0 \leqslant \lambda_+$, while if $\lambda_+ > 0$ then $\lambda_{\min} \leqslant 0 < \lambda_+$.

Since the function $L(\lambda)$ is strictly increasing on $(\lambda_{\min}, \lambda_+)$, in that interval there always exists a unique solution $L^{(-1)}(t)$ to the equation $L(\lambda) = t$ for $t \in (L(\lambda_{\min}), L(\lambda_+))$. Set

$$\lambda_q := L^{(-1)}(-q) \in (\lambda_{\min}, \lambda_+), \qquad -q \in (L(\lambda_{\min}), L(\lambda_+)).$$

Now introduce the "Cramér's transform" of ξ as a random variable $\xi^{(\lambda)}$ with the "tilted" distribution $\mathbf{P}(\xi^{(\lambda)} \in dt) = \dfrac{e^{\lambda t}}{\varphi(\lambda)} \mathbf{P}(\xi \in dt)$. It is clear that

$$L'(\lambda) = \frac{\varphi'(\lambda)}{\varphi(\lambda)} = \mathbf{E}\, \xi^{(\lambda)}, \qquad \lambda \in (\lambda_-, \lambda_+),$$

so that $\mu_q := \mathbf{E}\, \xi^{(\lambda_q)} \equiv L'(\lambda_q) > 0$.

Similarly to notation (2.1), for a numerical sequence $\{b_n\}$ we will denote by $\tilde{b}_n$ its "moving averages" over intervals of lengths $d(n)$:

$$\tilde{b}_n := \frac{1}{d(n)} \sum_{n \leqslant k < n + d(n)} b_k, \qquad n \geqslant 1.$$

Theorem 4.1 *Assume that* $\lambda_+ - \lambda_- > 0$ *and* $a_n = b_n e^{qn}$, *where* $\{b_n\}$ *satisfies condition* $[\boldsymbol{\psi}, \uparrow \cup \downarrow]$ *with* $\psi(t) = \sqrt{t}$, *and* $-q \in (L(\lambda_{\min}), L(\lambda_+))$.

(i) *In the non-lattice case, for any fixed* $0 < \Delta_1 < \Delta_2 < \infty$, *the relation*

$$h(x, \Delta) \sim \frac{(1 - e^{-\lambda_q \Delta}) e^{-\lambda_q x}}{\mu_q \lambda_q} \tilde{b}_{x/\mu_q}, \qquad x \to \infty, \tag{4.1}$$

holds uniformly in $\Delta \in [\Delta_1, \Delta_2]$. *Here, if* $q = 0$ *then* $\lambda_q = 0$, $\mu_q = \mu$ *and, by continuity, the coefficient of* $\tilde{b}_{x/\mu_q}$ *in* (4.1) *turns into* Δ/μ, *as in* (2.10).

(ii) *In the arithmetic max-span 1 case, for integer-valued $x \to \infty$ one has*

$$\sum_{n=0}^{\infty} a_n \mathbf{P}(S_n = x) \sim \frac{e^{-\lambda_q x}}{\mu_q} \, \tilde{b}_{x/\mu_q}. \tag{4.2}$$

Proof. We will only present the proof in the non-lattice case; in the arithmetic one, the argument is even simpler. Using the standard observation that, for the sum $S_n^{(\lambda)} := \xi_1^{(\lambda)} + \cdots + \xi_n^{(\lambda)}$ of n i.i.d. copies of $\xi^{(\lambda)}$, one has $\mathbf{P}(S_n^{(\lambda)} \in dt) = \dfrac{e^{\lambda t}}{\varphi^n(\lambda)} \mathbf{P}(S_n \in dt)$ (this was also used in Example 1 in [20]), we obtain

$$e^{qn}\mathbf{P}(S_n \in dt) = e^{(q+L(\lambda))n} e^{-\lambda t} \mathbf{P}(S_n^{(\lambda)} \in dt).$$

Therefore, splitting $\Delta[x)$ into K subintervals $\Delta_x[x + j\Delta_x)$, $j = 0, 1, \ldots, K-1$, of length $\Delta_x := \Delta/K$, where $K = K(x) \to \infty$ slowly enough, we have

$$\begin{aligned}
h(x, \Delta) &= \sum_n b_n e^{qn} \mathbf{P}(S_n \in \Delta[x)) = \sum_n b_n \int_{\Delta[x)} e^{qn} \mathbf{P}(S_n \in dt) \\
&= \sum_n b_n \int_{\Delta[x)} e^{-\lambda_q t} \mathbf{P}(S_n^{(\lambda_q)} \in dt) \\
&= \sum_n b_n \sum_{j=0}^{K-1} \int_{\Delta_x[x+j\Delta_x)} e^{-\lambda_q t} \mathbf{P}(S_n^{(\lambda_q)} \in dt) \\
&\sim \sum_n b_n \sum_{j=0}^{K-1} e^{-\lambda_q (x + j\Delta_x)} \mathbf{P}(S_n^{(\lambda_q)} \in \Delta_x[x + j\Delta_x)) \\
&= \sum_{j=0}^{K-1} e^{-\lambda_q (x + j\Delta_x)} \sum_n b_n \mathbf{P}(S_n^{(\lambda_q)} \in \Delta_x[x + j\Delta_x)),
\end{aligned} \tag{4.3}$$

where the change of summation order in the last line can be justified using relation (4.4) below and condition $[\boldsymbol{\psi}, \uparrow \cup \downarrow]$.

Now note that, for the inner sum on the right-hand side of (4.3), one has

$$\sum_n b_n \mathbf{P}(S_n^{(\lambda_q)} \in \Delta_x[x + j\Delta_x)) \sim \frac{\Delta_x}{\mu_q} \, \tilde{b}_{(x+j\Delta_x)/\mu_q} \sim \frac{\Delta_x}{\mu_q} \, \tilde{b}_{x/\mu_q}, \quad x \to \infty, \tag{4.4}$$

uniformly in $j \leqslant K$ (the second equivalence holds since $\{\tilde{b}_n\}$ is ψ-l.c. for $\psi(t) = \sqrt{t}$). Indeed, if $\{b_n\}$ satisfies condition $[\boldsymbol{\psi}, \downarrow]$, the relation follows from an argument similar to the one proving assertion (i) of Theorem 2.1, but with a_n replaced by b_n, and S_n replaced by $S_n^{(\lambda_q)}$: the results of Lemmata 2.1, 2.2 and 2.5(i) are still applicable in this case, so one only needs to show that, for the "left-most sum" Σ_{2-} (now with $m_- = x/(r\mu_q)$), one has $\Sigma_{2-} = o(\tilde{b}_{x/\mu_q})$. This can be easily done using the exponential

Chebyshev's inequality, the obvious representation $\mathbf{E}\, e^{\delta\xi^{(\lambda_q)}} = \varphi(\lambda_q+\delta)/\varphi(\lambda_q)$ (for $\delta \in (0, \lambda_+ - \lambda_q)$) and the fact that, by Corollary 1 in [6], one has $\tilde{b}_n = e^{o(\sqrt{n})}$. Namely, choosing $\eta := (r-1)/3 > 0$ and observing that $\varphi(\lambda_q+\delta)/\varphi(\lambda_q) \leqslant e^{\delta\mu_q(1+\eta)}$ for small enough $\delta > 0$, we obtain that

$$\begin{aligned}
\Sigma_{2-} &= \sum_{n<m_-} b_n \mathbf{P}(S_n^{(\lambda_q)} \in \Delta_x[x + j\Delta_x)) \preccurlyeq \sum_{n<m_-} \tilde{b}_n \mathbf{P}(S_n^{(\lambda_q)} \geqslant x) \\
&\leqslant \sum_{n<m_-} e^{o(\sqrt{n})-\delta x} \left(\frac{\varphi(\lambda_q+\delta)}{\varphi(\lambda_q)} \right)^n \\
&\leqslant \sum_{n<m_-} \exp\{o(\sqrt{n}) - \delta x + n\delta\mu_q(1+\eta)\} \\
&\preccurlyeq \sum_{n<m_-} \exp\{-\delta x + n\delta\mu_q(1+2\eta)\} \\
&\preccurlyeq \exp\{-\delta x + m_-\delta\mu_q(1+2\eta)\} = e^{-\delta\eta x/r},
\end{aligned}$$

which completes the proof of (4.4) since, as we said above, $\tilde{b}_{x/\mu_q} = e^{o(\sqrt{x})}$. The case where $\{b_n\}$ satisfies condition $[\boldsymbol{\psi}, \uparrow]$ is dealt with in a similar way.

Thus we showed that the expression in the last line in (4.3) is equal to

$$(1+o(1))\frac{\tilde{b}_{x/\mu_q}}{\mu_q} \sum_{j=0}^{K-1} e^{-\lambda_q(x+j\Delta_x)}\Delta_x \sim \frac{\tilde{b}_{x/\mu_q}}{\mu_q} \int_{\Delta[x)} e^{-\lambda_q t} dt = \frac{(1-e^{-\lambda_q\Delta})e^{-\lambda_q x}}{\mu_q\lambda_q}\, \tilde{b}_{x/\mu_q},$$

which establishes (4.1). In the arithmetic case, the proof proceeds in the same way. Theorem 4.1 is proved. $\square$

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